

Theoretical Steady State Distribution of an Additive in Sugar Beet Diffusers

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Introduction of an appropriate chemical into a diffuser may be considered for any of several purposes such as the suppression of microbiological action or the control of texture of the pulp. The additive may be introduced at various points in the diffuser either intermittently or in an essentially continuous manner. This paper considers the question of how the additive becomes distributed in the various portions of the diffuser and between the juice and pulp leaving the diffuser. Provided certain assumptions hold, this distribution can be simply expressed for the steady state reached in the case of a diffusion battery consisting of discrete cells to one of which a fixed amount of additive is introduced after each transfer step. Following presentation and discussion of the results for the steady state distribution of additive in such a model battery, the modifications required for similar batteries operating in accord with less restrictive assumptions and for continuous diffusers are discussed. The time required to reach a steady state is also briefly discussed. All mathematical derivations are contained in the appendix.

Model Battery

Let the cells of the battery be numbered serially. The cosettes enter the first cell and leave the last cell as pulp, while the battery supply water enters the last cell and leaves the first cell as juice. In this paper a model battery is defined as one for which the following assumptions hold:

1. Each cell contains the same volume of cosettes, V_c , and the same volume of juice, V_j . At each transfer the contents of each cell are completely replaced, the volume V_c of cosettes of any particular cell moving to the next higher numbered cell and the volume V_j of juice moving to the next lower numbered cell.

2. The additive present in each cell immediately after a transfer becomes distributed so that immediately prior to the next transfer the ratio of the amount of additive associated with the juice to the amount of additive associated with the cosettes is the same in each cell, viz. the partition ratio d .

3. No additive is consumed or produced by reactions taking place in any cell.

When steady state conditions are attained in an n -celled model battery to the m^{th} cell of which the amount of additive a is intro-

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duced after each transfer, the amount of the additive which leaves the battery with the pulp at each transfer is

$$P = \left(\frac{d^m - 1}{d^{n+1} - 1} \right) a$$

and the amount which leaves the battery with the juice is

$$J = a - P = \left(\frac{d^{n+1} - d^m}{d^{n+1} - 1} \right) a$$

Under steady state conditions the amount of additive in the i^{th} cell is

$$A_i = \left(\frac{d+1}{d-1} \right) J (1-d^{-i}) \quad \text{for } i = 1 \text{ to } m,$$

and

$$A_i = \left(\frac{d+1}{d-1} \right) P (d^{n+1-i} - 1) \quad \text{for } i = m \text{ to } n.$$

The total amount of additive in the battery is then

$$T = \left(\frac{d+1}{d-1} \right) [ma - (n+1) P]$$

Evaluation of these expressions for d equal to unity gives

$$P = \frac{ma}{n+1},$$

$$A_i = 2iJ \quad \text{for } i = 1 \text{ to } m,$$

$$A_i = 2(n+1-i) P \quad \text{for } i = m \text{ to } n,$$

and

$$T = (n+1-m) ma.$$

Derivation of these results is given in the appendix.

Examination of these mathematical expressions leads to several generalizations applying to steady state conditions in a model battery to which the same amount of additive is introduced in a particular cell following each transfer:

1. Regardless of its position, the cell into which the additive is introduced (the add cell) contains more additive than any other cell.

2. For any specified value of the partition ratio d , the amount of additive in the add cell is the same when it is the $(n-m)^{\text{th}}$ cell as when it is the m^{th} cell, regardless of the value of m .

3. For any specified value of d , the add cell contains more additive the closer it is located to the center of the battery.

4. Regardless of its position, the add cell contains more additive the closer d is to unity.

5. For a battery operated so that d is unity, the additive is distributed linearly between the add cell and each end cell.

6. The further d is from unity, the more non-linear is the distribution of additive between the add cell and each end cell. When the partition ratio is greater than unity this non-linearity is in the direction of more additive than linear distribution for cells ahead of the add cell (i less than m), and less for cells behind the add cell. This non-linearity is in the opposite direction for d less than unity.

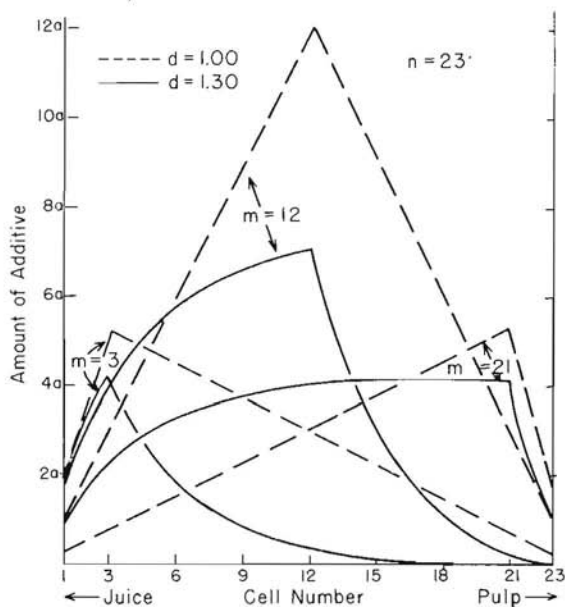


Figure 1.—Steady state distribution of additive in a 23-cell model battery for two values of the partition ratio d and three positions (m) of the add cell. Quantities shown are the steady state amounts of additive in each cell where a is the amount introduced in the add cell after each transfer.

These generalizations are illustrated in Figure 1 where the calculated steady state distributions are shown for a 23-cell battery for two values of d and for three different positions of the add cell. Although drawn as smooth curves or straight lines in the figure, these results have significance only for the points corresponding to integral cell numbers, the ordinate being the amount of additive present at the steady state in the cell number shown as abscissa. For example, when the add cell is at the center of the battery ($m = 12$), it contains $12a$ units of additive at the steady state for a partition coefficient of unity ($d = 1.00$). It is at once apparent from the figure that the amount of additive in a cell may be many times or a small fraction of the amount

added after each transfer depending on the value of d and the position of the cell with respect to the add cell.

If it is desired to maintain in all cells a concentration of additive greater than some specified minimum value, the correct choice of add cell is that for which the steady state amounts of additive in the first and last cells of the battery are most nearly the same. This location moves from the center of the battery for $d = 1$ progressively nearer the pulp end as d is increased above unity. For the case $n = 23$, $d = 1.30$, the correct choice of feed cell for this purpose is $m = 21$. In general its location is given by the integral value of m nearest that for which $d^{m-1} = \frac{d^n + 1}{d - 1}$, and $\frac{a}{P(1+d)}$ or $\frac{da}{J(1+d)}$, whichever is larger, is the amount of additive which must be introduced after each transfer to maintain at least a units of additive in each cell.

The proportion of additive leaving a model battery in the

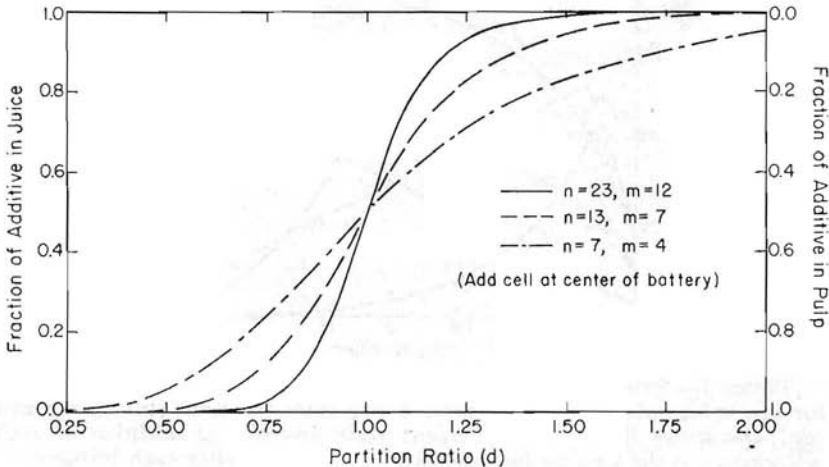


Figure 2.—Dependence on the partition ratio (d) of the steady state fractions of additive appearing in the juice and pulp when the add cell is at the center of model batteries containing 7, 13, and 23 cells.

juice depends on the number of cells in the battery, the value of the partition ratio, and the location of the add cell. The dependence on n and d is shown for steady state conditions in Figure 2 for introduction of the additive at the center of the battery. Either an increase in the number of cells or a shift of d value away from unity leads to a preponderance of additive being discharged from the juice end of the battery for d greater than unity and from the pulp end for d less than one. The effect of the position of the add cell on the emergence of the additive

from the battery is shown for selected values of d for a 23-cell model battery at steady state conditions in Figure 3. Thus for $d = 1.30$, the proportion of additive leaving the 23-celled battery in the juice may be varied from 23 to 99.94% by varying the location of the feed cell.

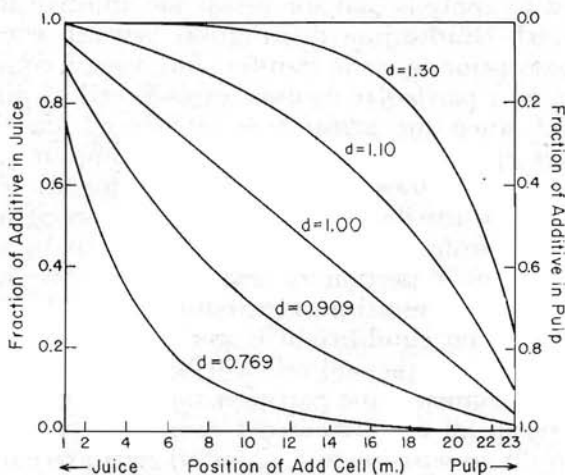


Figure 3.—Dependence on the position of the add cell of the steady state fractions of additive appearing in the juice and pulp of a 23-cell model battery for five different values of the partition ratio (d).

Some Non-Model Batteries

The expressions given above are also useful in describing batteries which deviate in certain ways from satisfying assumptions 1 and 2 for a model battery. First, there may not be a clean separation of the juice from the cossettes when a transfer is made. Provided the same fraction of the juice phase is transported with the cossettes from each cell at transfer and the assumptions made for the model battery are otherwise valid, the above equations are still applicable, the only change being that the partition ratio then has a different value than if a clean separation were made. This is demonstrated in the appendix.

Another situation described by the model battery equations is that where some of the contents of each cell are allowed to remain in the cell at each transfer. Provided the relative amounts of additive associated with the portions transferred in each direction and with the portion remaining are the same for each cell at transfer, and provided the model battery assumptions are otherwise valid, the expressions given for P and J are still applicable. The expressions for the amount of additive in each cell and the total amount of additive in the battery are each increased by the factor $(1+a)$ where a is the ratio of the amount

of additive remaining in the cell to that which is removed from the cell at each transfer. These statements are also verified in the appendix.

A third case for which the results for a model battery can be modified to apply is that for which the additive in each cell does not reach equilibrium distribution between the juice and cossette phase prior to each transfer, but where equilibrium is approached in a particular manner in each cell as discussed below. Except when the additive is introduced via the battery supply water or the fresh cossettes, the assumption for a model battery that the partition ratio is the same for all cells implies that equilibrium distribution of additive is reached in each cell prior to each transfer. This can be readily seen by considering the movement of a portion of additive after its introduction into cell m . If the additive is introduced into the juice phase of the add cell and equilibrium is not attained before transfer, the partition ratio in the add cell will be greater than the equilibrium value. Similarly the partition ratio will exceed its equilibrium value in all cells where net transfer of additive is from the juice to the cossette phase, i.e., for all cells preceding the add cell, or for i less than m . On the other hand net transfer of additive is from the cossettes to the juice for cells after the add cell, hence the partition ratio for these cells will be less than the equilibrium value if equilibrium is not attained. Thus for all cells to have the same partition ratio, equilibrium must be reached since this ratio is approached from opposite directions in cells before and after the add cell. For the special case where equilibrium distribution of additive is not attained in each cell, but where one partition ratio (D) applies to all cells preceding the add cell and another partition ratio (d) to all cells after the

add cell, $P = \frac{a}{1+X}$ where $X = \frac{(d^{n+1}-m-1) D^m (D-1)}{(d-1) (D^m-1)}$. The ad-

ditive contents of the individual cells are given by the expressions for the model battery with the appropriate value of D or d in place of d . The effect of additive distribution failing to reach equilibrium prior to transfer when additive is introduced into the juice phase is to increase the proportion of additive leaving the battery in the juice as compared to its value if distribution equilibrium had been attained.

Continuous Countercurrent Diffuser

The continuous countercurrent diffuser corresponding to the model battery defined above may be thought of as a model battery in which the number of cells is increased indefinitely and

the individual cell volume is proportionately decreased. Such a diffuser is referred to here as an ideal continuous countercurrent diffuser. In practice, however, the second assumption for a model battery cannot hold for the infinitely small cells thus hypothesized for the ideal continuous diffuser. It was shown above that this assumption is equivalent to attaining distribution equilibrium in each cell prior to transfer; in the case of the ideal continuous diffuser this is equivalent to having the oppositely moving streams of juice and cossettes in equilibrium with respect to distribution of additive at every point. Since such equilibrium cannot be established instantaneously in practice, this ideal continuous diffuser is of little interest.

In the theory of partition chromatography and distillation theory, the concept of equivalent theoretical plates has proved useful. For the present problem, we define an equivalent theoretical cell as a portion of a continuous countercurrent diffuser such that the concentration of additive associated with the juice leaving one end of the cell is that corresponding to distribution equilibrium with the concentration of additive associated with the cossettes entering the other end of the cell. The volume of an equivalent theoretical cell will be smaller the more intimate the mixing of the cossettes and juice as they travel in opposite directions and the lower their relative velocities. We continue to retain the assumptions made above for the model battery, with modifications appropriate for the continuous diffuser. The first assumption now means that the continuous diffuser has uniform cross section made up of the same area of juice and cossettes at all points with the rates of flow of each phase uniform throughout the diffuser. It is further assumed that the juice and cossettes each have a uniform concentration of additive over any cross section of the diffuser. In place of a single partition ratio applicable to all cells, a single distribution coefficient is assumed to apply to the equilibrium distribution of additive between cossettes and juice over the entire ranges of composition occurring in the continuous diffuser. The distribution coefficient c is defined as the ratio of the concentration of additive in juice to that in cossettes when the additive has reached distribution equilibrium between the two phases. A continuous countercurrent diffuser for which the above assumptions hold is referred to hereafter as a model continuous diffuser.

In order to illustrate the concept of equivalent theoretical cells and assist in its application to model continuous diffusers, let us examine qualitatively the case where equilibrium distribution corresponds to equal concentrations of additive in the juice

and the cossettes. Consider a model continuous diffuser of length λ in which juice flows from right to left (from $x = \lambda$ to $x = 0$), cossettes flow from left to right (from $x = 0$ to $x = \lambda$), and additive is introduced continuously into the juice at $x = \mu$. Let j_x and s_x be the concentration of additive at position x for the juice and cossette phases, respectively, and c the distribution coefficient, i.e. the equilibrium distribution value of j/s . Then the fact that equilibrium distribution of additive between the juice and cossettes is not attained instantaneously means that j_x/s_x is greater than c to the left of the add point and less than c to the right of the add point; in other words it must change sharply close to the add point. We also know this ratio to be infinite at the left end and zero at the right end of the diffuser when neither the battery supply water nor the fresh cossettes contain any additive. Thus qualitatively we have the situation shown in Figures 4 and 5 respectively for the general appearance of the ratio j_x/s_x and for j_x and s_x individually as a function of position in the diffuser. Both j_x and s_x are shown in Figure 5 as varying linearly with position in anticipation of results similar to those for model batteries with partition ratio of unity. The ratio j_x/s_x shown in Figure 4 corresponds to the concentrations shown in Figure 5. If back diffusion of additive in the juice is negligible, j is expected to show a sharp break at the add point.

Let us now place one theoretical cell boundary immediately to the right of the add point such that s has essentially its maximum value and j has its maximum value to the right of the

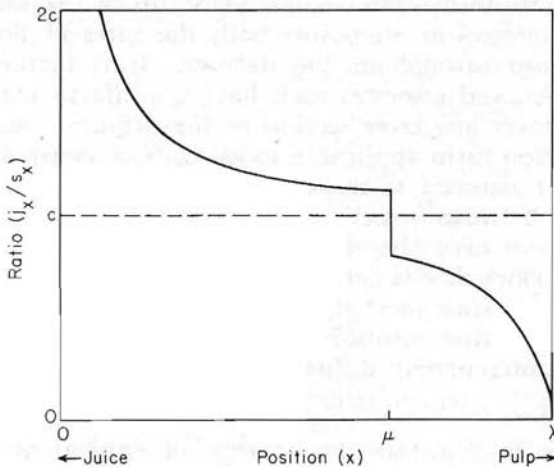


Figure 4.—Qualitative dependence of the steady state ratio of concentrations of additive in juice and cossettes on position in a model continuous countercurrent diffuser. Add point is at $x = \mu$.

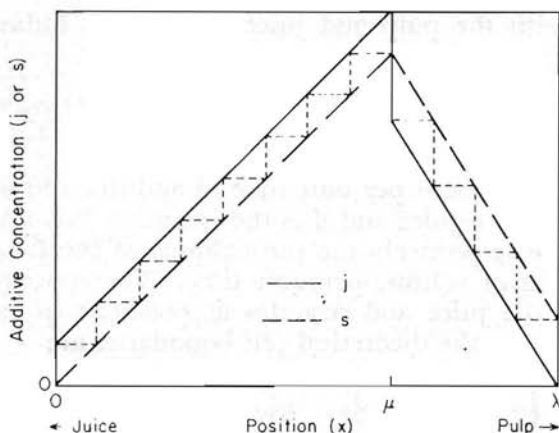


Figure 5.—Variation of steady state additive concentration in juice and cosettes in a model continuous countercurrent diffuser with partition ratio of unity. Add point is at $x = \mu$. Division of the diffuser into equivalent theoretical cells is indicated.

sharp break at the add point. Additional cell boundaries are then determined such that $j_{i-1}/s_i = c$ where j_{i-1} is the concentration of additive in the juice leaving the i^{th} cell and s_i is the concentration in the cosettes leaving the i^{th} cell. The locations of the boundaries thus determined are designated by the vertical dotted lines for the example of Figure 5. At steady state conditions, the distance along the axis of the diffuser corresponding to a theoretical cell is thus seen to be the distance separating the points in the juice and cosettes at which the additive concentrations have the same ratio as they would have at equilibrium. In Figure 5 this distance is shown as uniform along the length of the diffuser. It is shown in the appendix that all theoretical cells on either side of the add point have the same volume provided the distribution coefficient is constant. However, this volume may be different on the two sides of the add point if net transfer of additive between the juice and cosettes takes place more readily in one direction than the other. This latter situation is equivalent to a battery in which two different partition ratios apply respectively to cells before and after the add cell.

Let n designate the number of theoretical cells numbered serially from left to right such that m designates the theoretical add cell. Expressions for steady state conditions for the model continuous diffuser are then found to be similar to those for model batteries given above. Thus the concentrations of additive

associated with the pulp and juice leaving the diffuser are respectively

$$P = \frac{a}{u_s} \left(\frac{d^m - 1}{d^{n+1} - 1} \right) \text{ and } J = \frac{a}{u_j} \left(\frac{d^{n+1} - d^m}{d^{n+1} - 1} \right)$$

where a is the amount per unit time of additive introduced continuously into the juice and d is the quantity ($u_j c / u_s$) in which u_j and u_s are respectively the throughputs of the juice and cosettes in units of volume per unit time. The concentrations of additive in the juice and cosettes at positions in the diffuser corresponding to the theoretical cell boundaries are

$$j_i = \left(\frac{d^{i+1} - 1}{d - 1} \right) \frac{J}{d^i}, \quad s_{i+1} = j_i / c \quad (i = 0, \text{ to } m-1)$$

$$\text{and } s_i = \left(\frac{d^{n+1-i} - 1}{d - 1} \right) P, \quad j_{i-1} = c s_i \quad (i = m \text{ to } n)$$

where $i-1$ and i refer to the left and right hand boundaries of the i^{th} cell. The total amount of additive in the diffuser at the steady state is approximately

$$T = \frac{V_s}{n u_s} \left(\frac{d'+1}{d-1} \right) [ma - (n+1) u_s P] + V_s P \left(\frac{d'}{d} - 1 \right)$$

where d' is the quantity ($V_j c / V_s$) and where V_j and V_s are respectively the volumes of juice and cosettes in the diffuser. It will be noted that this expression reduces to one similar to that for the model battery if $d' = d$, which is true when the throughput of each phase is proportional to the volume occupied by that phase in the diffuser at the steady state. Evaluation of these expressions for d equal to unity gives

$$u_s P = \frac{ma}{n+1}, \quad u_j J = \frac{(n+1-m)a}{n+1};$$

$$j_i = (i+1) J, \quad s_{i+1} = j_i c \quad (i = 0 \text{ to } m-1);$$

$$s_i = (n+1-i) P, \quad j_{i-1} = c s_i \quad (i = m \text{ to } n);$$

$$\text{and } T = \frac{(d'+1)V_s}{2u_s} (n+1-m) ma + \frac{V_s P}{2} (d'-1).$$

The expressions given for T assume that all theoretical cells have the same volume. Derivation of these results is given in the appendix.

Relation to the Theory of Fractionation by Countercurrent Distribution

If instead of adding a single substance, a mixture of two substances were added, and if the partition coefficient of one were greater than unity and that of the other less than unity, Figure 3 indicates that these two substances might be efficiently separated with one leaving the battery in the juice and the other in the pulp. This is an example of separating two substances by countercurrent distribution, a technique which has been vigorously exploited in recent years. The distribution of additive in sugar beet diffusion batteries is thus seen to be a particular case of the more general problem of distribution of a substance between two immiscible phases moving countercurrent to each other. The steady state relations presented above are essentially the same as those which have been developed and applied in countercurrent extraction theory (1).

Time to Reach Steady State Conditions

Up to this point nothing has been said concerning how long is required to reach steady state conditions after introduction of additive is begun. Since steady state conditions are approached asymptotically, strictly speaking an infinite time is required. However some information on the approximate time required to approach steady state conditions can be obtained by considering the manner in which a single shot of additive becomes distributed in a battery subsequent to its introduction. Since this problem is related to the development of a band of a substance on a partition chromatographic column or in the tubes of a countercurrent distribution apparatus, the theory developed for those situations (2, 3, 4) can be applied to the present problem. After a number of transfers have been carried out subsequent to introduction of a single shot of additive, the latter becomes distributed among the various cells such that at any time there is a maximum amount in one cell with smaller amounts in other cells, the amount in any cell being less the farther the cell is from the cell containing the maximum. Except when the partition ratio is unity, the location of the cell containing the maximum moves further from the add cell as the number of transfers is increased, or the band is said to move away from the add point. The movement is toward the juice end for the partition ratio greater than unity. The number of transfers required to move this maximum out of a model battery is

$$t = \frac{m(d+1)}{(d-1)} \quad \text{if } d \text{ is greater than unity}$$

and $t = \frac{(n-m)(1+d)}{(1-d)}$ if d is less than unity. The correspond-

ing expressions for the time required to move the band of additive to the juice or pulp ends of a model continuous diffuser

are respectively $t = \frac{\mu V_s(d'+1)}{u_s(d-1)}$ if d exceeds unity or $t =$

$\frac{(\lambda-\mu)V_s(1+d')}{u_s(1-d)}$ when d is less than one. Symbols here have the

same meanings as before. Further details are given in the appendix.

One may think of the distribution of additive in the diffuser at any time as the sum of the distributions of each portion of additive already introduced or the superposition of the bands due to successive additions. Since the number of transfers or time calculated by the above expressions represents the shortest period of operation to insure movement of the first band maximum out of the diffuser, it may be considered a minimum period for approaching steady state conditions. Each additional transfer or equivalent transfer time then results in an additional band maximum leaving the diffuser. If the number of transfers or time calculated by the above expressions is much larger than the number of cells or the mean throughput time of the juice and cosettes, one or two times this period may serve as a practical estimate of how long is required to reach steady state conditions. If such is not the case, several times the mean throughput time is suggested as a better estimate. The theory has been developed for quantitatively expressing the approach to steady state conditions (6), but is cumbersome to apply.

Both the shape of the band due to a single shot of additive and its maximum value as it leaves the battery or diffuser are related to the partition ratio d and the number of (theoretical) cells, the band becoming more spread out and the maximum becoming smaller the larger t . Although the expressions relating band maximum and band width to partition ratio and number of theoretical plates in partition chromatography are relatively simple, the corresponding relationships have not been developed in a form satisfactory for application to the diffuser problem considered here.

Discussion

The theoretical results presented above are probably most useful as a basis for understanding and predicting the manner in which a substance which is constantly introduced into a sugar beet diffuser becomes distributed within the diffuser and between the diffusion juice and the pulp. The effect of the partition ratio, the number of cells, and the location of the add point on the distribution has been discussed in conjunction with Figures 1, 2, and 3.

The mathematical expressions for the steady state presented above apply quantitatively only if the assumptions made in their derivation are known to be valid. Despite the fact that any beneficial effect of an additive presumably results from some chemical reaction, the assumption that there is no consumption of additive in the diffuser may be essentially true. Its validity can be ascertained either by direct measurement of the steady state additive content in the juice and pulp or by separate experiment. The design and method of operation of the diffuser will normally indicate if the assumption with regard to the constancy of the relative volume of juice and cossettes in the different portions of the diffuser is satisfied. In the case of the model battery, it was further assumed that the effective partition ratio is the same for all cells. It was pointed out that this implied that sufficient time was allowed between transfers for equilibrium distribution to be reached in each cell. This condition is not likely to be fulfilled in commercial practice with Roberts batteries, but it may be fairly closely approached. A single effective partition ratio for all cells also implies the applicability of a single distribution coefficient c for all parts of the diffuser. Distribution coefficients are in general dependent on the composition of the phases involved, the temperature, and the concentrations of the substance distributed. Whether the variations with respect to these properties in various parts of the diffuser produce significant variations in c must be considered for each additive of interest.

Use of the model battery steady state expressions to estimate the amount of additive in the diffusion juice, the pulp, or the various cells requires a knowledge of the partition ratio d . This is probably best determined directly under actual or simulated operating conditions. Direct measurement is also the best check on the constancy of d for different cells, particularly for cells before and after the add point. Of course d can be estimated by measuring any of the steady state quantities mentioned above if the pertinent assumptions are known to hold. If practically all

additive leaving the diffuser is associated with either the juice or the cossettes, this in effect places respectively a lower or upper limit on the effective partition ratios estimated from the additive content of the diffusion juice or the pulp.

For model continuous diffusers, u_j , u_s , V_j , V_s , and a are presumably known. This leaves d , n , and m as unknown quantities appearing in the steady state expressions presented above. However, since $d = u_j c / u_s$, it is a knowledge of c which is required. The distribution coefficient c is best found from separate equilibrium distribution measurements, but it can also be estimated from the time required for maximum additive concentration to leave the diffuser after a single shot has been introduced at a known position in accordance with the relation already discussed. There is apparently no simple way to estimate m and n independently of the steady state expressions. By measuring J or P and assuming that all theoretical cells have the same size on both sides of the add point ($m/n = \mu/\lambda$), one can then estimate n and m from the appropriate steady state expression. These values would be minimum estimates if the flow of additive from the diffuser were essentially all from one end. If it is feasible to determine steady state concentrations of additive at selected points within the diffuser, the size of the theoretical cells can be determined from such measurements. Another approach to the problem is to determine m by steady state measurements when additive is introduced into the battery supply water and determine $(n-m)$ from similar measurements when additive is introduced with the fresh cossettes. Further discussion of these methods is given in the appendix.

If some additive from the pulp water is introduced with the battery supply water in addition to its introduction at another point in the diffuser, the steady state expressions applying are simply the sum of the appropriate expressions for supplying additive from each source separately.

Measurements have been made (5) on the lactic acid content of cell juices from Roberts batteries and continuous diffusers in which various patterns of distribution of acid among the cells were found. Although fermentation in more than one cell may have been responsible for these observed patterns, it is also clear that they could be approximated in model batteries such as discussed above with lactic acid generation occurring in only one cell.

Acknowledgment

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Appendix

Symbols used:

- n = number of cells in the battery or number of equivalent theoretical cells in a continuous countercurrent diffuser.
- i = integral subscript index running from 1 to n and designating the individual cells of a battery numbered serially with juice leaving the battery at $i = 1$, pulp at $i = n$. For continuous countercurrent diffusers i runs from 0 to n and designates the boundaries of the theoretical cells with juice leaving at $i = 0$ and pulp at $i = n$.
- m = serial number of the add cell.
- L_p, S_i = amounts of additive associated with the juice or the cosettes respectively in the i th cell of a battery immediately prior to the next transfer under steady state conditions.
- j_i, s_i = concentrations of additive in the juice or cosettes respectively at the indicated theoretical cell boundary in a continuous countercurrent diffuser at the steady state.
- u_j, u_s = rate of movement of the juice or cosettes respectively through a continuous diffuser in units of volume per unit time.
- V_j, V_s = total continuous diffuser volume occupied by juice or cosettes respectively.
- a = quantity of additive introduced into add cell of a battery after each transfer or amount per unit time of additive introduced continuously in a continuous diffuser at the add point.

c = distribution coefficient = equilibrium value of j/s .

d = partition ratio. For batteries, $d = L_i/S_i$ immediately prior to the next transfer under steady state conditions. For continuous diffusers,

$$d = \frac{u_j c}{u_s}$$

$r = 1/d$.

J, P = amount of additive leaving the battery with the juice or the pulp respectively for each transfer under steady state conditions; for continuous diffusers the concentration of additive in the diffusion juice or pulp under steady state conditions.

A_i = total amount of additive in the i^{th} cell of a battery or the i^{th} theoretical cell of a continuous diffuser at the steady state.

T = total amount of additive in the diffuser under steady state conditions.

$$d' = \frac{V_f c}{V_s}$$

x = position in continuous countercurrent diffuser with $x = 0$ being the juice end, $x = \lambda$ the pulp end, and $x = \mu$ the add point.

The following sections supply further details in reference to portions of the text under the same headings.

Model Battery

Transfer under steady state conditions must satisfy the following material balance equation for each cell except the first, last, and feed cells:

$$L_i + S_i = L_{i+1} + S_{i-1} \quad (i = 2 \text{ to } m-1; m+1 \text{ to } n-1) \quad (1)$$

The following equations similarly apply to the end cells:

$$S_1 + J = L_2 \quad (2)$$

$$P + L_n = S_{n-1} \quad (3)$$

Since $S_i = r L_i$, equations (1) and (2) can be rewritten to give

$$L_{i+1} = (1+r) L_i - r L_{i-1} \quad (i = 2 \text{ to } m-1) \quad (4)$$

$$L_2 = (1+r) J \quad (5)$$

Putting $i = 2$ in equation (4) and making use of equation (5), we find

$$L_3 = (1+r+r^2) J \quad (6)$$

Similarly putting $i = 3$ in equation (4) and making use of equations (5) and (6), we find

$$L_4 = (1+r+r^2+r^3) J \quad (7)$$

Repeating this process we can find expressions for L_i for all values of i from 1 to m , viz.

$$L_i = (1+r+r^2+\dots+r^{i-1}) J \quad (8)$$

The sum in parentheses may be expressed compactly by making use of the identity:

$$\sum_{i=0}^k z^i = \frac{z^{k+1}-1}{z-1} \quad (9)$$

This then gives the expression

$$L_i = \left(\frac{r^i-1}{r-1} \right) J \quad (i = 1 \text{ to } m) \quad (10)$$

Since $S_i = rL_i$ and $A_i = L_i + S_i$, we have the following expression for the total amount of additive in the i^{th} cell:

$$A_i = (1+r) \left(\frac{r^i - 1}{r-1} \right) J = \left(\frac{d+1}{d-1} \right) (1-d^{-i}) J \quad (i = 1 \text{ to } m) \quad (11)$$

Corresponding results are similarly obtained for the last $n-m$ cells of the battery. Since $L_i = dS_i$, equations (1) and (3) can be rewritten to give

$$S_{i-1} = (1+d) S_i - d S_{i+1} \quad (i = m+1 \text{ to } n-1) \quad (12)$$

$$S_{n-1} = (1+d) P \quad (13)$$

Proceeding as above, we find that

$$S_i = \left(\frac{d^{n+1-i} - 1}{d-1} \right) P \quad (i = m \text{ to } n) \quad (14)$$

$$\text{and } A_i = \left(\frac{d+1}{d-1} \right) (d^{n+1-i} - 1) P \quad (i = m \text{ to } n) \quad (15)$$

Since both equations (11) and (15) are valid for the add cell, we can equate these two expressions for A_m :

$$\left(\frac{d+1}{d-1} \right) (1-d^{-m}) J = \left(\frac{d-1}{d+1} \right) (d^{n+1-m} - 1) P \quad (16)$$

Replacing J by $a-P$ and solving for P , we find

$$P = \left(\frac{d^m - 1}{d^{n+1} - 1} \right) a \quad (17)$$

$$J = a - P = \left(\frac{d^{n+1} - d^m}{d^{n+1} - 1} \right) a \quad (18)$$

The total amount of additive in the battery is then

$$T = \sum_{i=1}^n A_i = \sum_{i=1}^{m-1} \left(\frac{r+1}{r-1} \right) J (r^i - 1) + \sum_{i=m}^n \left(\frac{d+1}{d-1} \right) P (d^{n+1-i} - 1) \quad (19)$$

which can be written

$$T = \left(\frac{1+r}{1-r} \right) J \left(m - \sum_{i=0}^{m-1} r^i \right) + \left(\frac{d+1}{d-1} \right) P \left(m - n - 2 + \sum_{j=0}^{n+1-m} d^j \right) \quad (20)$$

When these sums are expressed as in equation (9) and J and r are replaced by $a-P$ and $1/d$ respectively, equation (20) simplifies to

$$T = \left(\frac{d+1}{d-1} \right) [ma - (n+1) P] \quad (21)$$

When $d = 1$, it is readily seen from equation (8) that

$$A_i = 2iJ \quad (i = 1 \text{ to } m) \quad (22)$$

Similarly the expanded form of equation (15) is seen to give

$$A_i = 2(n+1-i) P \quad (i = m \text{ to } n) \quad (23)$$

To evaluate P when $d = 1$, both the numerator and denominator of equation (17) are divided by $(d-1)$ and expressed as a summation using equation (9). This yields for $d = 1$

$$P = \frac{ma}{n+1}. \quad (24)$$

The total additive in the battery is then readily found to be

$$T = ma(n-m+1). \quad (25)$$

Some Non-Model Batteries

Suppose the juice and cosettes are not cleanly separated at transfer but the proportions of juice accompanying the cosettes and vice versa are each the same for all transfers. Let $L_i/S_i = d$ for clean separation and $L_i'/S_i' = D$ for this case. If the fraction of juice transferred with the cosettes is f_1 and the fraction of cosettes transferred with the juice is f_2 ,

$$D = \frac{L_i'}{S_i'} = \frac{(1-f_1)L_i + f_2S_i}{f_1L_i + (1-f_2)S_i} = \frac{(1-f_1)d + f_2}{f_1d + (1-f_2)} \quad (26)$$

Thus the net effect is to change the partition ratio from d to the value shown for D . All expressions for the model battery apply when this value of the partition ratio is used.

Suppose some of the contents of each cell are allowed to remain in the cell at each transfer but the material transferred in either direction and the residue are always in the same proportion. Again let $L_i/S_i = d$ for clean separation and $L_i'/S_i' = D$ for this case. If the fractions of juice and cosettes allowed to remain as residue are f_j and f_c respectively,

$$D = \frac{L_i'}{S_i'} = \frac{(1-f_j)L_i}{(1-f_c)S_i} = \frac{(1-f_j)d}{(1-f_c)} \quad (27)$$

Again the net effect is to change the partition ratio from d to the value shown for D . The expressions for P , J , L_i , and S_i for the model battery still apply when the appropriate partition ratio is used. If a is the ratio of the amount of additive left in a cell to the amount removed at each transfer, it is clear that the model battery expressions for the amount of additive in each cell and for the total amount of additive in the battery must be increased by the factor $(1+a)$. In terms of the symbols used here,

$$1 + a = \frac{D+1-D f_c - f_j}{(D+1)(1-f_j)(1-f_c)} \quad (28)$$

For the case where partition ratio D applies to the first m cells and partition ratio d applies to the last $m-n$ cells, equations (10), (11), (14), and (15) apply where $r = 1/D$ and $d = d$. When expressions for A_m of equations (11) and (15) are equated as in equation (16) and P and J are solved for as in Equations (17) and (18), the result is

$$P = \frac{a}{1+X} \quad \text{and} \quad J = \frac{aX}{1+X} \quad (29)$$

where X has the value given in the text.

Continuous Countercurrent Diffuser

Under steady state conditions the rate at which additive enters each cell is balanced by the rate at which it leaves. This yields a set of material balance equations similar to Equations (1), (2), and (3), viz.

$$j_i u_j + s_{i+1} u_s = j_{i+1} u_j + s_i u_s \quad (i = 1 \text{ to } m-2; m \text{ to } n-2) \quad (30)$$

$$j u_j + s_1 u_s = j_1 u_j \quad (31)$$

$$j_{m-1} u_j + P u_s = s_m u_s \quad (32)$$

The same procedure used in deriving the model battery equations yields

$$j_i = \left(\frac{r^{i+1}-1}{r-1} \right) J, \quad (i = 0 \text{ to } m-1) \quad (33)$$

$$s_i = \frac{(d^{n+1}-i-1)}{(d-1)} P \quad (i = m \text{ to } n) \quad (34)$$

The additive contents of the cassettes and juice leaving each theoretical cell are related by $j_{i-1} = c s_i$ ($i = 1$ to n). In particular $s_m = j_{m-1}/c$, so that equations (33) and (34) can each be used to express s_m . When these two expressions for s_m are equated and the relation $a = u_j/j + u_s P$ is used, the results are

$$P = \frac{a}{u_s} \left(\frac{d^{n+1}-1}{d^{m+1}-1} \right) \text{ and } J = \frac{a}{u_j} \left(\frac{d^{n+1}-d^m}{d^{m+1}-1} \right) \quad (35)$$

The total amount of additive in the diffuser at the steady state is estimated by Simpson's rule for the case where all theoretical cells of the diffuser have the same volume. For each theoretical cell

$$A_i = \frac{V_j}{2n} (j_{i-1} + j_i) + \frac{V_s}{2n} (s_{i-1} + s_i) \quad (36)$$

except for the add cell where

$$A_m = \frac{V_j}{2n} (j_{m-1} + j_{\max}) + \frac{V_s}{2n} (s_{m-1} + s_m) \quad (37)$$

Here j_{\max} is estimated from equation (33) with $i = m$. When the values for j and s given by equations (33) and (34) are inserted and the summation $T = \sum_{i=1}^n A_i$ is carried out in a manner similar to that used in obtaining equation (21) the result is

$$T = \frac{V_s}{n u_s} \left(\frac{d+1}{d-1} \right) [m a + (n+1) u_s P] + V_s P \left(\frac{d^r}{d} - 1 \right) \quad (38)$$

The corresponding relations when the partition ratio is unity are found in the same way as for the model battery to be those given in the text.

Evaluation of the above expressions for j , s , J , P , and T for countercurrent continuous diffusers with partition ratio of unity is carried out in a manner similar to that used for the model battery with $d = 1$. The results are given in the text.

To verify that all theoretical cells on the same side of the add point have the same volume, let us assume for the moment that this is true. Then equations (33) and (34) can be written

$$j_x = \left(\frac{r^x + 1 - 1}{r - 1} \right) J \text{ and } s_x = j_{(x-1)} / c \quad (39)$$

to express the concentration of additive in the juice and cossettes at any point between $x = 0$ and $x = \mu$. Where x is expressed in units of μ/m . Since these relations satisfy the differential materials balance relation

$$u_j \left(\frac{dj_x}{dx} \right) = u_s \left(\frac{ds_x}{dx} \right), \text{ the assumption of equal theoretical cell volumes is}$$

validated for the portion of the diffuser to which the relations used for j_x and s_x apply. In the same way the theoretical cells on the other side of the add point are shown to each have the same volume, but this volume is not necessarily the same on the two sides of the add point.

Time to Reach Steady State Conditions

The distribution of additive in each cell of a model battery prior to the next transfer can be described as fraction p in the juice and fraction

$$q = 1 - p \text{ in the cossettes, where } p = \frac{d}{1+d} \text{ and } q = \frac{1}{1+d}. \text{ Each transfer}$$

tends to move fraction p of the additive in each cell in one direction and fraction q in the other. For a single portion of additive introduced into the add cell, the effect of succeeding transfers will be to move additive toward the juice end of the battery p times as fast as the juice and toward the pulp end q times as fast as the cossettes. The net rate of movement of the maximum additive concentration will be toward the juice end if p is larger than q at an average of $p - q$ cells per transfer. Thus to move the

maximum a distance of m cells requires $m/(p - q) = m \left(\frac{d+1}{d-1} \right)$ transfers.

Likewise if the partition ratio is less than unity, $\frac{(n-m)}{(q-p)} = (n-m) \cdot \frac{1+d}{1-d}$ transfers are required to move the maximum out of the pulp end of the battery. In the continuous countercurrent diffuser, equilibrium distribution of additive between juice and cossettes corresponds to fraction

$$p = \frac{V_j j}{V_j + V_s s} = \frac{d'}{d'+1} \text{ in the juice and fraction } q = 1 - p = \frac{1}{d'+1} \text{ in the}$$

cossettes. The linear velocities of the juice and cossettes are $\frac{u_j}{V_j}$ and $\frac{u_s}{V_s}$ respectively. The net rate of movement of the maximum concentration of

additive due to a single injection of additive is accordingly $p \frac{u_j}{V_j} - q \frac{u_s}{V_s}$

The time necessary to travel to the juice end of the diffuser if p is greater than q is then

$$t = \frac{\mu}{p \frac{u_j}{V_j} + q \frac{u_s}{V_s}} = \frac{\mu V_s (d^t + 1)}{u_s (d^t - 1)} \quad (40)$$

Likewise if d is less than unity, the time required for the band to travel to the pulp end is

$$t = \frac{(\lambda - \mu) V_s (1 + d^t)}{u_s (1 - d)} \quad (41)$$

Discussion

If measurements of additive concentration at the steady state can be made at selected points within a model continuous diffuser, the equivalent theoretical cell volumes can be found by use of equation (39) and the similar equation applying for the other side of the add point. Suppose j_y and j_z is the measured steady state concentration of additive in the juice at points $x = y$ and $x = z$ between $x = 0$ and $x = \mu$. If $z/y = R$, from equation (39)

$$\frac{j_y}{j_z} = \frac{r^{y+1} - 1}{r^{Ry+1} - 1} \quad (42)$$

It is assumed that r has been determined. The value of y satisfying equation (42) then gives the number of theoretical cells between $x = 0$ and $x = y$, and hence the volume of each cell. If $J = j_a$ is used as j_z , only one measurement is needed within the diffuser. A similar procedure is used to determine theoretical cell volume for the other side of the add point.

If additive is introduced with the supply water instead of at the normal add point, y and z of equation (42) may be taken as the two ends of the diffuser. Then $J/a = \frac{r^{m+1} - 1}{r - 1}$, from which n_j is found. The volume of each theoretical cell on the juice discharge side of the normal add point is then the total diffuser volume divided by n_j . Similarly if additive is introduced with the fresh cossettes instead of at the normal add point, $p/a = \frac{d^{n_s+1} - 1}{d - 1}$, from which n_s is found. The volume of each theoretical cell on the pulp discharge side of the normal add point is then the total diffuser volume divided by n_s .